

# On random walks in random scenery

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**Abstract:** This paper considers 1-dimensional generalized random walks in random scenery. That is, the steps of the walk are generated by an arbitrary stationary process, and also the scenery is a priori arbitrary stationary. Under an ergodicity condition—which is satisfied in the classical case—a simple proof of the distinguishability of periodic sceneries is given.

## 1. Introduction

Random walks in random scenery have been studied by Mike Keane for quite some time (see [2] for his most recent work). In fact, he and Frank den Hollander were pioneers in this exciting area. Around 1985 they formulated a conjecture about “recovery of the scene” by a simple random walker. A weaker form of this: “distinguishability of two scenes”, was proven by Benjamini and Kesten ([1]). Since then there has been a lot of action in this field, especially by Matzinger and co-workers. We just mention the recent paper [5].

In the following we will introduce generalized random walks in random scenery, and analyse them from a dynamical point of view. This gives us in Section 2 a general scenery recovery result on the level of measures, from which we deduce in a simple way in Section 3 a proof for the distinguishability of periodic sceneries.

We consider a random walker on the integers. The integers are coloured by colours from an alphabet  $C$ . This is the scenery. At time  $n$  the walker records the scenery at his position, this yields  $r_n$  from  $C$ . To formalize somewhat more, let the random walk be described by a measure  $\mu$  on the Borel sets of  $\Omega$ , where

$$\Omega = \{\omega = (\omega_n)_{n \in \mathbb{Z}} : \omega_n \in J \text{ for all } n\}.$$

Here the set  $J$  of the possible steps of the walk will simply be  $\{-1, +1\}$ , or somewhat more general  $\{-1, 0, +1\}$ . Although often a single scenery  $x = (x_k)_{k \in \mathbb{Z}}$  is considered, it is useful to consider  $x$  as an element of the shift space  $X = C^{\mathbb{Z}}$  with shift map  $T : X \rightarrow X$ , equipped with some ergodic  $T$ -invariant measure  $\lambda$ , which we will call the *scenery measure*. We then consider  $x$  picked according to the measure  $\lambda$ . The colour record  $\varphi_x$  of  $x$  can be written as a map  $\varphi_x : \Omega \rightarrow X$ :

$$\varphi_x(\omega) = (r_n(\omega, x))_{n \in \mathbb{Z}},$$

where in line with the description above, one has for  $n \geq 1$

$$r_n(\omega, x) = (T^{\omega_0 + \dots + \omega_{n-1}} x)_0.$$

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This definition is completed by putting  $r_0(\omega, x) = x_0$ , and for  $n < 0$ :

$$r_n(\omega, x) = (T^{-\omega_{-1}-\omega_{-2}\cdots-\omega_n}x)_0.$$

The dynamics of the whole process is well described by a skew product transformation  $T_{\Omega \times X}$  on the product space  $\Omega \times X$  defined by

$$T_{\Omega \times X}(\omega, x) = (\sigma\omega, T^{\omega_0}x),$$

where  $\sigma$  denotes the shift map on  $\Omega$ .

Let us now look at the colour records of *all*  $x$ ; we define the *global recording map*  $\Phi : \Omega \times X \rightarrow X$  by

$$\Phi(\omega, x) = (r_n(\omega, x))_{n \in \mathbb{Z}}.$$

**Lemma 1.** *The map  $\Phi$  is equivariant, that is,  $\Phi \circ T_{\Omega \times X} = T \circ \Phi$ .*

*Proof.* One way:

$$\begin{aligned} \Phi \circ T_{\Omega \times X}(\omega, x) &= \Phi((\sigma\omega, T^{\omega_0}x)) = (r_n(\sigma\omega, T^{\omega_0}x))_{n \in \mathbb{Z}} \\ &= ((T^{\omega_1+\cdots+\omega_n}T^{\omega_0}x)_0)_{n \in \mathbb{Z}} = (r_{n+1}(\omega, x))_{n \in \mathbb{Z}}. \end{aligned}$$

The other way:

$$T \circ \Phi(\omega, x) = T((r_n(\omega, x))_{n \in \mathbb{Z}}) = (r_{n+1}(\omega, x))_{n \in \mathbb{Z}}. \quad \square$$

Clearly product measure  $\mu \times \lambda$  is preserved by  $T_{\Omega \times X}$ . We will be particularly interested in the image of  $\mu \times \lambda$  under the global recording map  $\Phi$ , which we denote  $\rho$ :

$$\rho = (\mu \times \lambda) \circ \Phi^{-1}.$$

We call  $\rho$  the *global record measure*. It follows from Lemma 1 that  $\rho$  is invariant for  $T$ . Moreover,  $\rho$  will be ergodic when  $T_{\Omega \times X}$  is ergodic for  $\mu \times \lambda$ . In the classical case where  $\mu$  is product measure this is guaranteed by Kakutani's random ergodic theorem. In this case, when  $\lambda$  and  $\lambda'$  are two scenery measures, and  $\rho = (\mu \times \lambda) \circ \Phi^{-1}$  and  $\rho' = (\mu \times \lambda') \circ \Phi^{-1}$  are the corresponding global record measures, then either  $\rho = \rho'$  or  $\rho \perp \rho'$ .

The colour record  $\varphi_x$  of a scenery  $x$  induces the *record measure*  $\rho_x$  defined by

$$\rho_x = \mu \circ \varphi_x^{-1}.$$

Following [4] we call the two sceneries  $x$  and  $y$  *distinguishable* if  $\rho_x \perp \rho_y$ . The following lemma shows that global distinguishability carries over to local distinguishability.

**Lemma 2.** *Let  $\lambda$  and  $\lambda'$  be two scenery measures with corresponding global record measures  $\rho$  and  $\rho'$ . Then  $\rho \perp \rho'$  implies that  $\rho_x \perp \rho_y$  for  $\lambda \times \lambda'$  almost all  $(x, y)$ .*

*Proof.* By Fubini's theorem, and recalling that  $\Phi(\omega, x) = \varphi_x(\omega)$ ,

$$\begin{aligned} \rho(E) &= \int_X \int_{\Omega} 1_E \circ \Phi(\omega, x) d\mu(\omega) d\lambda(x) \\ &= \int_X \mu(\varphi_x^{-1}E) d\lambda(x). \end{aligned}$$

So  $\rho = \int \rho_x d\lambda(x)$ . Hence if  $E$  is a Borel set with the property that  $\rho(E) = 1$  and  $\rho'(E^c) = 1$ , then  $\rho_x(E) = 1$  for  $\lambda$ -almost  $x$ , and  $\rho_y(E^c) = 1$  for  $\lambda'$ -almost  $y$ .  $\square$

## 2. Reconstructing the scenery measure

Here we consider the case of a generalized random walk with steps  $J = \{-1, 0, +1\}$  given by an ergodic stationary measure  $\mu$  on  $\Omega = J^{\mathbb{Z}}$ . For ease of notation we rename  $J$  to  $\{L, H, R\}$ . To simplify the exposition we will assume that there is no holding ( $\mu([H]) = 0$ ), and show at the appropriate moment that this restriction can trivially be removed.

There is a basic difference between symmetric walks and asymmetric walks in the reconstruction of the scenery. We call  $\mu$  *symmetric* if for each word  $w$

$$\mu[w] = \mu[\bar{w}].$$

Here  $\bar{w}$  denote the mirror image of  $w$ , that is, the word obtained from  $w$  by replacing R by L and L by R. Since he does not know left from right, a symmetric walker can only reconstruct a scenery  $x$  up to a reflection. This will result in two theorems, one for the asymmetric, and one for the symmetric case.

Let  $\lambda$  be a scenery measure. We give a few examples of calculation of the  $\rho$ -probabilities of cylinder sets. Let  $\mathcal{W}_n$  be the set of all words  $w = w_1 \dots w_n$  of length  $n$  over the (colour) alphabet  $\{0, 1\}$ . For  $w \in \mathcal{W}_n$  we let  $[w]$  denote the cylinder

$$[w] = \{x \in X : x_0 \dots x_{n-1} = w\},$$

and we will abbreviate  $\rho([w])$  to  $\rho[w]$ . We will use the same type of notations and conventions for  $\lambda$  and  $\mu$ . It is clear, using the stationarity of  $\lambda$ , that for instance

$$\rho[001] = \mu[RR]\lambda[001] + \mu[LL]\lambda[100],$$

and slightly more involved

$$\rho[000] = (\mu[RL] + \mu[LR])\lambda[00] + (\mu[RR] + \mu[LL])\lambda[000],$$

and

$$\rho[0001] = \mu[RLL]\lambda[100] + \mu[LRR]\lambda[001] + \mu[RRR]\lambda[0001] + \mu[LLL]\lambda[1000].$$

In the sequel we shall denote the word  $R \dots R$ ,  $N$  times repeated, as  $R^N$ . Note how with each appearance of a word  $w$  on the right side also the reversed word  $\bar{w}$  appears, defined by  $\bar{w} = w_n \dots w_1$  if  $w = w_1 \dots w_n$ . Words  $w$  that satisfy  $w = \bar{w}$  are called *palindromes*. Now let us put all the words  $w$  from  $\cup_{1 \leq k \leq n} \mathcal{W}_k$  in some fixed order, taking care that their lengths are non-decreasing and that for a fixed  $k$  we first take all palindromes, and then all non-palindromes in pairs  $(w, \bar{w})$ . Let  $V_n(\rho)$  and  $V_n(\lambda)$  denote the vectors of length  $(2^{n+1} - 2)$  containing the real numbers  $\rho[w]$  respectively  $\lambda[w]$  in the chosen order. For example,

$$V_2^T(\rho) = (\rho[0], \rho[1], \rho[00], \rho[11], \rho[01], \rho[10]).$$

In general, if  $w$  is a word of length  $N + 1$ , then  $\rho[w]$  is obtained as a sum of products  $\mu[u]\lambda[v]$ , where the length of  $v$  is *at most*  $N + 1$ , and length  $N + 1$  only occurs when the walker makes no turns, i.e., when  $u = R^N$  or  $u = L^N$ . Moreover, if  $w$  is a palindrome, then there is one maximal length term  $(\mu[R^N] + \mu[L^N])\lambda[w]$ , and if  $w$  is not a palindrome, then there are two maximal length terms  $\mu[R^N]\lambda[w]$ , and  $\mu[L^N]\lambda[\bar{w}]$ . This observation shows that there exists an almost lower triangular  $(2^{n+1} - 2) \times (2^{n+1} - 2)$  matrix  $A_n$  such that

$$V_n(\rho) = A_n V_n(\lambda).$$

Here ‘almost lower triangular’ means that  $A_n$  has the form

$$\begin{pmatrix} \square & 0 & 0 & 0 & 0 & 0 \\ * & \square & 0 & 0 & 0 & 0 \\ * & * & \square & 0 & 0 & 0 \\ * & * & * & \dots & 0 & 0 \\ * & * & * & * & \square & 0 \\ * & * & * & * & * & \square \end{pmatrix},$$

where (at palindrome entries)  $\square$  is a  $1 \times 1$  matrix  $\mu[R^N] + \mu[L^N]$ , and (at non-palindrome pairs)  $\square$  is a  $2 \times 2$  matrix of the form

$$\begin{pmatrix} \mu[R^N] & \mu[L^N] \\ \mu[L^N] & \mu[R^N] \end{pmatrix}.$$

With simple linear algebra we find that  $A_n$  is non-singular if and only if

$$\mu[R] \neq \mu[L], \dots, \mu[R^N] \neq \mu[L^N], \dots$$

Let us call a generalized random walk given by  $\mu$  *strongly asymmetric* if all these inequalities hold. For instance, if  $\mu$  is a stationary Markov chain given by a  $2 \times 2$  transition matrix  $(p_{s,s})$ , then  $\mu$  is strongly asymmetric if and only if  $p_{R,R} \neq p_{L,L}$ .

Note that when  $\mu[H] > 0$ , then only some *sub*-diagonal elements of  $A_n$  will change from 0 to a positive value. We therefore obtained the following result.

**Theorem 1.** *For strongly asymmetric generalized random walk with holding the scenery measure  $\lambda$  can be reconstructed from  $\rho$ .*

What remains is the symmetric walker case. Then in general  $\lambda$  can not be reconstructed from  $\rho$ . However, often we can reconstruct the *reversal symmetrized measure*  $\check{\lambda}$  defined for each word  $w$  by

$$\check{\lambda}[w] = \frac{1}{2}(\lambda[w] + \lambda[\bar{w}]).$$

For symmetric  $\mu$  the equation for, e.g.,  $\rho[0001]$  becomes

$$\rho[0001] = 2\mu[LRR]\check{\lambda}[001] + 2\mu[RRR]\check{\lambda}[0001].$$

In general, if  $w$  is a word of length  $N+1$ , then  $\rho[w]$  is obtained as a sum of products  $\mu[u]\check{\lambda}[v]$ , where the length of  $v$  is *at most*  $N+1$ , and length  $N+1$  only occurs when  $u = R^N$  or  $u = L^N$ . Moreover, now there is for all words  $w$  one term  $2\mu[R^N]\check{\lambda}[w]$  for the  $v = w$  with maximal length. So this time we obtain the existence of a  $(2^{n+1} - 2) \times (2^{n+1} - 2)$  lower triangular matrix  $A_n$  such that

$$V_n(\rho) = A_n V_n(\check{\lambda}).$$

Let us call  $\mu$  *straightforward* if arbitrary long words of R’s have positive probability to appear. Then the diagonal elements of  $A_n$  are positive for each  $n$ , and we obtain the following.

**Theorem 2.** *For straightforward symmetric generalized random walk with holding the reversal symmetrized scenery measure  $\check{\lambda}$  can be reconstructed from  $\rho$ .*

### 3. Distinguishing periodic sceneries

In this section we shall derive more general results with more simple proofs than in [3]. It is shown there that for asymmetric simple random walk with holding any two periodic sceneries  $x$  and  $y$  which are not translates of each other can be distinguished by their scenery records, i.e.,  $\rho_x \perp \rho_y$ . Our result is

**Theorem 3.** *Any strongly asymmetric generalized random walk with holding can distinguish two periodic sceneries that are not translates of each other, provided that their global record measures are ergodic.*

*Proof.* Let us write  $x \overset{t}{\sim} y$  if  $x$  and  $y$  are translates of each other, i.e., for some  $k$  one has  $y = T^k x$ . Let  $\text{Per}(x)$  be the period of  $x$ , i.e.  $p = \text{Per}(x)$  is the smallest natural number such that  $T^p x = x$ . Let  $\lambda$  be the scenery measure generated by  $x$ , i.e., denoting point measure in  $z$  by  $\delta_z$ ,

$$\lambda = \frac{1}{\text{Per}(x)} \sum_{k=0}^{\text{Per}(x)-1} \delta_{T^k x}.$$

The scenery measure generated by  $y$  is denoted as  $\lambda'$ . Now suppose that  $\rho_x$  is not orthogonal to  $\rho_y$ . Then, since  $\lambda$  and  $\lambda'$  are discrete, it follows from Lemma 2 that also  $\rho$  and  $\rho'$  are not orthogonal. But since these measures are ergodic, they must be equal. From Theorem 1 it then follows that also  $\lambda = \lambda'$ . This implies that  $x \overset{t}{\sim} y$ , by the discreteness of  $\lambda$  and  $\lambda'$ . Indeed, equality of these measures yields that  $\delta_{T^k x} = \delta_{T^j y}$  for some  $k$  and  $j$ , and hence that  $x \overset{t}{\sim} y$ .  $\square$

For a symmetric (generalized) random walk it is impossible to distinguish a sequence  $x$  from its reflection  $\overleftarrow{x}$ , defined by  $\overleftarrow{x}_k = x_{-k}$ . So let us call  $x$  and  $y$  equivalent, and we denote  $x \sim y$ , if  $y$  can be obtained from  $x$  by translation and/or reflection.

**Theorem 4.** *Any straightforward symmetric generalized random walk with holding can distinguish two periodic sceneries that are not equivalent, provided that their global record measures are ergodic.*

*Proof.* The proof follows the same path as the proof of Theorem 3, using Theorem 2 instead of Theorem 1. The only other difference now is that the measure  $\check{\lambda}$  is a mixture of point measures in  $T^k x$  and in  $T^j \overleftarrow{x}$ . But then equality of  $\check{\lambda}$  and  $\check{\lambda}'$  implies that  $y$  must be a translate, or the reflection of a translate of  $x$ , i.e.,  $x \sim y$ .  $\square$

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